Solution for Problem E: Career Planning Problem

Kazuhiro Hosaka, University of Tokyo

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1 Problem

There are N students ($A = \{1, 2, \ldots, N\}$) and N jobs ($B = \{1, 2, \ldots, N\}$). We are going to assign a job to each student, so that at most one student is assigned to each job. Some students may be jobless. That is, an *arrangement* is a function $f : A \to B \cup \{jobless\}$ satisfying that for each $x \in B$, there is at most one $i \in A$ with $f(i) = x$.

Each student $i \in A$ has a preference list over some jobs: $p_{i,1}, p_{i,2}, \ldots, p_{i,M_i} \in B$, which means that student i prefers job $p_{i,1}$ to job $p_{i,2}$, prefers job $p_{i,2}$ to job $p_{i,3}$, ..., and prefers job p_{i,M_i-1} to job p_{i,M_i}. Student i cannot be assigned to a job which does not appear in this list. So a valid arrangement f should satisfy that $f(i) = p_{i,s}$ for some s or $f(i) = jobless$.

For any two arrangements f_0 and f_1, we say a student $i \in A$ *prefers* f_0 *to* f_1 if job $f_0(i)$ is better than job $f_1(i)$ for student i, i.e., $f_0(i) = p_{i,s_0}$ and $f_0(i) = p_{i,s_1}$ with $s_0 < s_1$, or $f_0(i) \neq jobless$ and $f_1(i) = jobless$. We say f_0 is more *popular* than f_1 if the number of students who prefer f_0 to f_1 exceeds the number of students who prefer f_1 to f_0. We call an arrangement f *happy* if there is no arrangement which is more popular than f. Given the preference lists, the task is to determine whether a happy arrangement exists or not.

The inputs to this problem satisfy that $1 \leq N \leq 1000$ and $1 \leq M_i \leq N$ for each i.

2 Algorithm

Definition 2.1. *A job $x \in B$ is called* good *if there exists a student i with $x = p_{i,1}$ (that is, a job is good if it is most favored by some student). Let C be the set of good jobs. A job $x \in B$ is called* bad *if it is not good.*

Definition 2.2. *Let $P_i := \{p_{i,1}, p_{i,2}, \ldots, p_{i,M_i}\}$. For a student i and two jobs x and y, we write $x >_i y$ if $x = p_{i,s}$ and $y = p_{i,t}$ with $s < t$, or $x \in P_i$ and $y \notin P_i$. We write $x <_i y$ if $y >_i x$.*

Definition 2.3. *For $i \in A$, let $Q_i := \{x \in B \setminus C \mid$ there does not exist $y \in B \setminus C$ such that $y >_i x\}$.*

Remark 2.4. *If $P_i \subseteq C$, then $Q_i = B \setminus C$. Otherwise, $Q_i = \{p_{i,s}\}$ where s is the minimum index such that $p_{i,s} \notin C$ (Note that $s > 1$ here). Thus Q_i is the set of bad jobs which are most favored by student i.*

We consider a bipartite graph $G = (V, E)$ with $2N$ vertices $V := \{a_1, a_2, \ldots, a_N\} \cup \{b_1, b_2, \ldots, b_N\}$ and edges $E := \{(a_i, b_{p_{i,1}}) \mid i \in A\} \cup \{(a_i, b_x) \mid i \in A,\, x \in Q_i\}$. We are going to prove the following result:

Theorem 2.5. *There exists a happy arrangement if and only if the bipartite graph G described above has a perfect matching.*

Constructing G can be obviously done in $O(N^2)$ time and $O(N^2)$ space. In order to find whether G has a perfect matching or not, we can apply Ford-Fulkerson algorithm, or faster, Hopcroft-Karp algorithm. The time complexity is bounded by $O(N^3)$ or $O(N^{2.5})$ respectively.

3 Correctness

We first note that the concept of being "jobless" can be removed. For an arrangement $f : A \to B \cup \{\textit{jobless}\}$, we can construct a bijection $\widetilde{f} : A \to B$ which agrees with f on $f^{-1}(B)$. f can be recovered from $\widetilde{f}$ as follows:

$$f(i) = \begin{cases} \widetilde{f}(i) & \text{if } \widetilde{f}(i) \in P_i \\ \textit{jobless} & \text{if } \widetilde{f}(i) \notin P_i \end{cases}$$

No matter how we choose the images of the elements of $f^{-1}(\{\textit{jobless}\})$, $\widetilde{f}$ will have exactly the same happiness as that of f. That is, for any two arrangements f and g, if we construct bijections $\widetilde{f}$ and $\widetilde{g}$ as above from f and g, respectively, then f is more popular than g if and only if $\widetilde{f}$ is more popular than $\widetilde{g}$. So from now on we modify our definition of arrangements.

Definition 3.1. *An* arrangement *is a bijection $f : A \to B$. For two arrangements f_0 and f_1, we say a student $i \in A$* prefers *f_0* to *f_1 if $f_0(i) >_i f_1(i)$. The definition of popularity and happiness are kept.*

Note that a perfect matching in G can be naturally considered as an arrangement. So when f denotes a perfect matching and (a_i, b_x) is in f, we regard f as a bijection from A to B and use the notation like $f(i) = x$.

Lemma 3.2. *Let f be a perfect matching in G. For $i \in A$, either $f(i) = p_{i,1}$ or $f(i) \in Q_i$.*

Proof. This follows immediately from the construction of the edge set E. □

Lemma 3.3. *Let f be a perfect matching in G, and g be an arrangement. For $i, j \in A$ and $x, y, z \in B$ with $f(i) = x$, $f(j) = g(i) = y$ and $g(j) = z$, if $x <_i y$, then $y >_j z$.*

Proof. By Lemma 3.2, either $x = p_{i,1}$ or $x \in Q_i$. $x = p_{i,1}$ contradicts to $x <_i y$, so we have $x \in Q_i$, and by the definition of Q_i, we have $y \in C$. Again by Lemma 3.2, either $y = p_{j,1}$ or $y \in Q_j$, therefore $y = p_{j,1}$. Since $x \neq y$, we have $i \neq j$ and so $y \neq z$. This proves the lemma. □

Lemma 3.4. *Let f be a perfect matching in G. Then f is a happy arrangement.*

Proof. Take any arrangement g. Then $f^{-1} \circ g$ is a permutation on A, so it decomposes into cyclic permutations.

Take one of those cycles $(i_1\, i_2 \cdots i_c)$ and let $i_{c+1} = i_1$ (here $i_{l+1} = (f^{-1} \circ g)(i_l)$). If a student i_l

in this cycle prefers g to f, then $f(i_l) <_{i_l} g(i_l) = f(i_{l+1})$. Now we can apply Lemma 3.3, to obtain $f(i_{l+1}) >_{i_{l+1}} g(i_{l+1})$, that is, student i_{l+1} prefers f to g. This observation gives the fact that the number of students who prefers g to f is less than or equal to the number of students who prefers f to g in this cycle.

By applying this argument for every cycle, we conclude that g is not more popular than f. Thus f is a happy arrangement. □

Now the "if" part of Theorem 2.5 has been proven. Here goes the proof of the "only if" part.

Lemma 3.5. *Let f be an arrangement. For a student $i \in A$, assume that $x = f(i) \in C$. If there exists a job $y \in B$ such that $x <_i y$, then f is not a happy arrangement.*

Proof. Let $j = f^{-1}(y) \, (\neq i)$. Since $x \in C$, there exists $k \in A$ such that $x = p_{k,1}$. Taking such k, we have $k \neq i$ because $x <_i y$.

If $k = j$, we can make an arrangement g such that $g(i) = y$, $g(j) = x$, and $g(h) = f(h)$ for any $h \in A \setminus \{i, j\}$. Since $g(i) = y >_i x = f(i)$ and $g(j) = x = p_{j,1} >_j y = f(j)$, there are two students who prefer g to f, i and j, and no students who prefer f to g. So g is more popular than f.

If $k \neq j$, we can make an arrangement g such that $g(i) = y$, $g(j) = f(k)$, $g(k) = x$, and $g(h) = f(h)$ for any $h \in A \setminus \{i, j, k\}$. Since $g(i) = y >_i x = f(i)$ and $g(k) = x = p_{k,1} >_k f(k)$, there are two students who prefer g to f, i and k, and at most one student who prefers f to g, possibly j. So g is more popular than f.

In either case we find an arrangement which is more popular than f, so f is not happy. □

Lemma 3.6. *Let f be an arrangement. For a student $i \in A$, assume that $x = f(i) \in B \setminus C$. If there exists a bad job $y \in B \setminus C$ such that $x <_i y$, then f is not a happy arrangement.*

Proof. Let $j = f^{-1}(y) \, (\neq i)$ and $z = p_{j,1}$. Since $z \in C$, we have $z \neq x$ and $z \neq y$. Then taking $k = f^{-1}(z)$, we have $k \neq i$ and $k \neq j$.

We can make an arrangement g such that $g(i) = y$, $g(j) = z$, $g(k) = x$, and $g(h) = f(h)$ for any $h \in A \setminus \{i, j, k\}$. Since $g(i) = y >_i x = f(i)$ and $g(j) = z = p_{j,1} >_j y = f(j)$, there are two students who prefer g to f, i and j, and at most one student who prefers f to g, possibly k. So g is more popular than f, and this shows that f is not happy. □

By Lemma 3.5 and Lemma 3.6, for a happy arrangement f and for each $i \in A$, if $f(i) \in C$ then $f(i)$ must be most favored, and otherwise $f(i)$ must be most favored among the bad jobs by student i. Therefore $(a_i, b_{f(i)}) \in E$, i.e., f is a perfect matching in G.

This completes the proof of Theorem 2.5.